

Analysis of Correlation Matrices Using Covariance Structure Models

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It is often assumed that covariance structure models can be arbitrarily applied to sample correlation matrices as readily as to sample covariance matrices. Although this is true in many cases and leads to an analysis that is mostly correct, it is not permissible for all structures. This article reviews three interrelated problems associated with the analysis of structural models using a matrix of sample correlations. Depending upon the model, applying a covariance structure to a matrix of correlations may (a) modify the model being studied, (b) produce incorrect values of the omnibus test statistic, or (c) yield incorrect standard errors. An important class of models are those that are scale invariant (Browne, 1982), for then Errors a and b cannot occur when a correlation matrix is analyzed. A number of examples based on restricted factor analysis are presented to illustrate the concepts described in the article.

In recent years the study of structural models for covariance matrices (Bentler, 1983; Jöreskog, 1978) has become increasingly important in behavioral research. The appeal of this method is that it provides a framework for translating general ideas about behavior into a more explicit quantitative description and encourages comparisons of the relative performance of two or more structures with empirical data.

Although theoretical work has been presented to justify the application of certain models to correlation matrices (Browne, 1977; McDonald, 1975; Shapiro & Browne, 1986), the only complete statistical theory for structural model analysis has been developed for covariance matrices (Browne, 1974; Jöreskog, 1970; Lawley & Maxwell, 1971). This contrasts with common practice in the behavioral sciences in which correlations are most often emphasized in data analyses. Apparently, it is not well understood that applying a covariance structure to a correlation matrix will produce some combination of incorrect test statistics, incorrect standard errors, or incorrect parameter estimates and may in fact alter the model being studied, unless the model under examination is appropriate for scale changes. To put the matter more directly, structural models cannot be arbitrarily applied to a matrix of correlations at the convenience of the researcher, even though it is natural and often quite important to do so. Kim and Feree (1981), for example, advocate analyses of standardized variables because behavioral science measurements have arbitrary scales, and because interpretation is facilitated thereby. Structures that are appropriate for standardized variables are therefore of considerable utility in practice.

In spite of several technical discussions of this problem (Bentler & Lee, 1983, section 1; Browne, 1982, section 1.2; Krane & McDonald, 1978; Lee & Fong, 1983; Shapiro & Browne, 1986; Swain, 1975; Swaminathan & Algina, 1978), incorrect applications of structural models to correlation matrices have persistently appeared¹ (Bagozzi, 1980, p. 140, hypotheses 1 & 2; Bentler & Lee, 1978, p. 348; Blok & Saris, 1983; Cole, Howard & Maxwell, 1981; Cornelius, Willis, Nesselroade, & Baltes, 1983; Cunningham, 1980, p. 141; Everitt, 1984, section 3.6.3; Jöreskog, 1981, section 4.4; Jöreskog & Sörbom, 1984, p. III.89; Lansman, Donaldson, Hunt, & Yantis, 1982, Table 9; Lee, 1979, section 4; Linn & Werts, 1982, Table 5.5; MacLeod, Jackson, & Palmer, 1986; Marsh & Hocevar, 1983, p. 241; McDonald, 1980, section 4, examples 1-3; Rindskopf, 1984, p. 41; Werts, Breland, Grandy, & Rock, 1980). It is possible, indeed likely, that erroneous theoretical conclusions have been reported because of this statistical misunderstanding.

The purpose of this article is to review the statistical issues associated with applications of structural models to correlation matrices. Inasmuch as descriptions of this problem have appeared mostly in technical sources, it seems appropriate to summarize it in a more generally accessible form. In the process of doing so, this review emphasizes models that are invariant with respect to changes in scale. Some suggestions are given for alternative ways to estimate a model when a correlation matrix is examined.

For convenience in this discussion, I make use of examples from restricted factor analysis. Restricted factor analysis is a specialization of more general covariance structural models, for example, the well-known latent variable model operationalized by the LISREL computer program (Jöreskog & Sörbom, 1984, version VI). It is, nonetheless, a very useful special case, in that it serves as a building block for other more complex structures.

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¹ Although this list is hardly exhaustive, it includes several distinguished scientists at the insistence of a referee who reassuringly stated, "It is good for one's character, not bad for it, to acknowledge past errors and clearly be capable of learning."

Indeed, one characterization of the LISREL model itself is as a system of three interdependent factor analysis structures (Jöreskog & Sörbom, 1977, section 2.1). Furthermore, much of the recent literature specifically concerned with the effects of analyzing correlation matrices with structural models has dealt with factor analysis. Finally, a large proportion of the applications of structural models in psychological research deal with factor models. For these reasons, this article emphasizes factor analysis models and attempts to generalize from them to other cases.

The actual use of many statistical methods in present-day behavioral research is often dependent upon the particular way in which one or two computer programs implement a general idea for data analysis. For most social scientists, the analysis of covariance structures is more or less synonymous with the program and mathematical model in LISREL, even though perhaps half a dozen other programs have been written and applied successfully. Although LISREL is an important development, it does not incorporate all of the features needed for the tremendous range of problems in applied research. Some useful models simply cannot be formulated as special cases of the particular structure operationalized in LISREL. Each of the other programs one might consider has a somewhat different set of limitations, however, so this observation is not specific to LISREL. Nonetheless, because of its wide availability, and because no other alternative is clearly preferable, this article assumes that LISREL is the program on hand. This imposes a particular philosophy of covariance structures that is not necessarily optimal for the problems described here; but to give a realistic review of the comparative advantages of alternative programs would complicate this review substantially and not benefit the reader enough to justify the additional material.

Scale-Invariant Models and Scale-Free Parameters

It will be useful in the following discussion to define certain terms and equivalence relations. A regrettable aspect of the literature on this topic is that there is little consistency in the labels applied to particular aspects of the problem. For example, the terms *scale-free*, *scale independent*, and *scale invariant* in unrestricted factor analysis each have been applied to the property that the factor pattern from a correlation matrix is simply a rescaling of the factor pattern from the associated covariance matrix. Consequently, in this discussion I use terms that are felt to be appropriate for the various aspects of equivalence that are reviewed and refer to specific definitions as needed. This, unfortunately, guarantees that the terms used here will not always coincide with the terms used elsewhere. In particular, I use labels that differ from those favored by Krane and McDonald (1978) in their important treatment of this subject, although several of the specific relationships were first emphasized in their article.

Definition. A model for a covariance matrix is a symmetric matrix-valued function $\Sigma = \Sigma(\gamma)$ of a parameter vector γ , where the order of Σ and γ are $p \times p$ and q , respectively.

For an illustration, consider three factor analysis models:

$$\Sigma_k = \Sigma(\gamma_k) = \Lambda_k \Phi_k \Lambda_k' + \Psi_k, \quad k = 1, 3, \tag{1}$$

where for Model 1,

$$\Lambda_1' = \begin{pmatrix} 1 & \lambda_1 & 0 & 0 \\ 0 & 0 & 1 & \lambda_2 \end{pmatrix}, \quad \Phi_1 = \begin{pmatrix} \sigma_1^2 & \sigma_{21} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix},$$

$$\Psi_1 = \text{diag}(\psi_1, \psi_2, \psi_3, \psi_4),$$

so that $\gamma_1' = (\lambda_1, \lambda_2, \sigma_1^2, \sigma_{21}, \sigma_2^2, \psi_1, \psi_2, \psi_3, \psi_4)$; for Model 2,

$$\Lambda_2' = \begin{pmatrix} \lambda_1 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_2 & \lambda_3 \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix},$$

$$\Psi_2 = \text{diag}(\psi_1, \psi_2, \psi_3, \psi_4),$$

so that $\gamma_2' = (\lambda_1, \lambda_2, \lambda_3, \rho, \psi_1, \psi_2, \psi_3, \psi_4)$; and for Model 3,

$$\Lambda_3' = \begin{pmatrix} \lambda_1 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & \lambda_4 \end{pmatrix}, \quad \Phi_3 = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix},$$

$$\Psi_3 = \text{diag}(\psi_1, \psi_2, \psi_3, \psi_4),$$

so that $\gamma_3' = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \rho, \psi_1, \psi_2, \psi_3, \psi_4)$.

Definition. A correlation structure is a symmetric matrix-valued function, $\mathbf{P} = \mathbf{P}(\theta)$, with diagonal elements equal to unity.

For factor analysis, one writes

$$\mathbf{P} = \mathbf{P}(\theta) = \dot{\Lambda} \dot{\Phi} \dot{\Lambda}' + \dot{\Psi},$$

and it is emphasized that $\text{diag}(\mathbf{P}) = \text{diag}(\dot{\Lambda} \dot{\Phi} \dot{\Lambda}' + \dot{\Psi}) = \mathbf{I}$.

Definition. A model $\Sigma(\gamma)$ is *scale invariant* if, for any diagonal matrix $\mathbf{D}_\alpha = (\alpha_j \neq 0)$ and any parameter vector γ , there exists another parameter vector γ^* such that

$$\Sigma(\gamma^*) = \mathbf{D}_\alpha \Sigma(\gamma) \mathbf{D}_\alpha \tag{2}$$

(Browne, 1982, section 1.2). Individual transformations for elements of γ^* in Equation 2 are represented by the functions

$$\gamma_j^* = f_j(\alpha, \gamma_j), \quad j = 1, \dots, q, \tag{3}$$

where $\alpha' = (\alpha_1, \dots, \alpha_p)$.

To illustrate, with the parameter vector

$$\gamma^{*'} = (\lambda_1^*, \lambda_2^*, \sigma_1^{*2}, \sigma_{21}^*, \sigma_2^{*2}, \psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*),$$

where

$$\lambda_1^* = (\alpha_2/\alpha_1)\lambda_1, \tag{4a}$$

$$\lambda_2^* = (\alpha_4/\alpha_3)\lambda_2, \tag{4b}$$

$$\sigma_1^{*2} = \alpha_1^2 \sigma_1^2, \tag{4c}$$

$$\sigma_{21}^* = \alpha_3 \alpha_1 \sigma_{21}, \tag{4d}$$

$$\sigma_2^{*2} = \alpha_3^2 \sigma_2^2, \tag{4e}$$

and

$$\Psi^{*'} = (\psi_j^*) = \mathbf{D}_\alpha^2 \Psi, \tag{4f}$$

it can be seen that Model 1 is scale invariant because

$$\begin{aligned} \Sigma(\gamma_1^*) &= \Lambda_1^* \Phi_1^* \Lambda_1^{*'} + \Psi_1^* = \begin{bmatrix} \sigma_1^{*2} + \psi_1^* & & & \\ \lambda_1^* \sigma_1^{*2} & \lambda_1^{*2} \sigma_1^{*2} + \psi_2^* & & \\ \sigma_{21}^* & \lambda_1^* \sigma_{21}^* & \sigma_2^{*2} + \psi_3^* & \\ \lambda_2^* \sigma_{21}^* & \lambda_1^* \lambda_2^* \sigma_{21}^* & \lambda_2^* \sigma_2^{*2} & \lambda_2^{*2} \sigma_2^{*2} + \psi_4^* \end{bmatrix} \\ &= \begin{bmatrix} \alpha_1^2(\sigma_1^2 + \psi_1) & & & \\ \alpha_2 \alpha_1 \lambda_1 \sigma_1^2 & \alpha_2^2(\lambda_1^2 \sigma_1^2 + \psi_2) & & \\ \alpha_3 \alpha_1 \sigma_{21} & \alpha_3 \alpha_2 \lambda_1 \sigma_{21} & \alpha_3^2(\sigma_2^2 + \psi_3) & \\ \alpha_4 \alpha_1 \lambda_2 \sigma_{21} & \alpha_4 \alpha_2 \lambda_1 \lambda_2 \sigma_{21} & \alpha_4 \alpha_3 \lambda_2 \sigma_2^2 & \alpha_4^2(\lambda_2^2 \sigma_2^2 + \psi_4) \end{bmatrix} = \mathbf{D}_\alpha \Sigma(\gamma_1) \mathbf{D}_\alpha. \end{aligned}$$

By the same argument, Model 3 also can be shown to be scale invariant. For Model 2, on the other hand, it is not possible to choose γ_2^* such that $\Sigma(\gamma_2^*) = \mathbf{D}_\alpha \Sigma(\gamma_2) \mathbf{D}_\alpha$, unless one also takes $\alpha_2 = \alpha_3$, which violates the requirement that components of \mathbf{D}_α be arbitrary.

An important corollary of the previous definition is that if a model $\Sigma(\gamma)$ is scale invariant, then there exists a parameter vector $\dot{\gamma}$ such that $\mathbf{I} = \text{diag} [\Sigma(\dot{\gamma})]$. Obviously, every covariance matrix can be transformed into the corresponding correlation matrix by rescaling, using $\mathbf{P} = \mathbf{D}_\sigma^{-1} \Sigma \mathbf{D}_\sigma^{-1}$, where $\mathbf{D}_\sigma = \text{diag} (\Sigma)^{1/2}$. Therefore, every scale-invariant covariance structure can be transformed into the associated correlation structure by simply rescaling the model parameters by functions of standard deviations. With Model 1, for example, one takes $\mathbf{D}_\sigma = \mathbf{D}_\sigma$ in Equation 2, with individual relations among parameters specified by Transformations 4. Conversely, a scale-invariant covariance structure can be developed from its associated correlation structure by the inverses of these transformations, assuming that standard deviations are known.

Definition. A parameter $\gamma_i \in \gamma$ is *scale-free* (a) if the model $\Sigma(\gamma)$ of which it is a part is scale invariant and (b) if for all choices of \mathbf{D}_α in Equation 2, $\gamma_i^* = \gamma_i$ (Krane & McDonald, 1978).

For example, there are no scale-free parameters in Model 1 because Condition b is not met for any γ_i . Model 2 is not scale invariant (Condition a). Model 3 is scale invariant, but the only scale-free parameter is the factor correlation ρ . All other parameters of Model 3 are scale dependent.

In summary, a structural model for a covariance or correlation matrix is a matrix-valued function in which the $p(p + 1)/2$ elements of Σ or \mathbf{P} are scalar-valued functions of q independent variables γ . A correlation structure further requires that $\text{diag} [\mathbf{P}(\gamma)] = \mathbf{I}$. Scale invariance is a property of a model such that any rescaling of the covariance matrix yields another covariance matrix that also satisfies the model. An individual parameter in a scale-invariant model is scale-free if it remains unchanged in all rescalings of the covariance matrix. A model that is scale-invariant may contain parameters that are scale-free.

Applying Covariance Structure Models to Correlation Matrices

When a covariance structure is applied to a matrix of observed score correlations, the circumstances of the analysis change considerably, although this condition may not be appre-

ciated. As a result, the analysis of correlation matrices is often associated with several kinds of errors, one or more of which may be present in any particular context. The difficulty of greatest concern is that the scaling of the sample moment matrix often alters the form of the structure being studied. A second problem concerns the possibility that different values of the omnibus test statistic may be produced when correlations rather than variances and covariances are examined. A third issue is that the estimated standard errors associated with individual parameter estimates are incorrect in an analysis of correlation matrices, unless certain specific corrections are made.

Effects of Scale Changes on the Model

Although factor analysis models such as Model 1 generally are developed for covariance matrices, it often is of interest to examine an associated correlation structure. In order to do so, however, one must explicitly consider the population standard deviations (Krane & McDonald, 1978, section 5), that is, one must consider a model of the form

$$\Sigma = \mathbf{D}_\sigma \Sigma^* \mathbf{D}_\sigma = \mathbf{D}_\sigma (\Lambda \Phi \Lambda' + \Psi) \mathbf{D}_\sigma, \tag{5}$$

where the diagonal matrix \mathbf{D}_σ contains scaling terms. If Structure 5 were to be studied in practice, it would entail estimating up to p more coefficients than Model 1, depending upon the identifiability of the particular model. It is important to note that $\text{diag} (\Sigma^*) = \mathbf{I}$ does not necessarily obtain, unless $\mathbf{D}_\sigma = \text{diag} (\Sigma)^{1/2}$. Nonetheless, to assume that the factor model is a correlation structure, the unit diagonal elements of Σ^* must obtain exactly. A model similar to Model 5 was studied by Browne (1982, pp. 108, 131; see also Bentler & Lee, 1983), who imposed the constraint

$$\text{diag} (\Sigma^*) = \text{diag} (\Lambda \Phi \Lambda' + \Psi) = \mathbf{I}, \tag{6}$$

by numerical methods to ensure that $\Sigma^* = \mathbf{P}$ is a correlation matrix. This method requires the use of special-purpose computer programs, however. LISREL in particular has not yet implemented constraints of this kind.

Many users of factor analysis proceed by standardizing the sample covariance matrix, \mathbf{S} , to the associated matrix, \mathbf{R} , of sample correlations and then applying Model 1 to the transformed data without regard for the possible effects this has on the structure. This may not affect the analysis in any unexpected way, depending upon the particular model. In other settings, the rescaling changes the model completely. For if the

model is not appropriate for such transformations, the structure actually fit is

$$\tilde{\Sigma} = \mathbf{D}_s(\Lambda\Phi\Lambda' + \Psi)\mathbf{D}_s, \quad (7)$$

where the diagonal matrix \mathbf{D}_s contains sample standard deviations. But \mathbf{D}_s is a stochastic matrix, varying from sample to sample. \mathbf{D}_s is an estimator of the population standard deviations but not of \mathbf{D}_σ under Model 5. Clearly, Models 7 and 1 are very different, inasmuch as the former contains the stochastic term \mathbf{D}_s , and the latter does not. Models 7 and 5 are also not equivalent, because the former assumes $\hat{\mathbf{D}}_\sigma = \mathbf{D}_s$, whereas in the latter, \mathbf{D}_σ is a parameter to be estimated.

This point can be generalized by summarizing as follows: it is not always possible to apply an arbitrary covariance structure such as

$$\Sigma = \Sigma(\gamma) \quad (8)$$

to a matrix of observed correlations without regard for the effect that such a transformation may impose on the model. By carrying out the analysis of the original model with correlations, one modifies the structure being studied, so that

$$\tilde{\Sigma} = \mathbf{D}_s\Sigma(\gamma)\mathbf{D}_s, \quad (9)$$

and Structures 8 and 9—the first of which the researcher believes is examined and the second of which is actually analyzed—are not identical.

Although this problem arises in general with the analysis of a correlation matrix, it does not occur with scale-invariant models. For example, if a factor analysis model is invariant, it is always possible to obtain estimates of the parameters such that the conditions in Equation 6 hold. In this case, it is permissible to apply the model to a matrix of correlations. The original structure will not be modified. The effect of rescaling \mathbf{S} to \mathbf{R} only modifies elements of the parameter vector γ .

For example, Model 1 and Model 3 are scale invariant and may be applied to a correlation matrix. In contrast, Model 2 can only be applied to a covariance matrix, unless special computing methods are used, because otherwise the actual model studied is Model 7, which of course is not the same as Model 1.

Rescaling Parameter Estimates From Scale-Invariant Models

Even if a model is scale-invariant, there is no guarantee that the reproduced matrix implied by the model will have a unit diagonal. According to the definition given earlier, however, correlation structures must have

$$\text{diag}(\hat{\mathbf{P}}) = \mathbf{I}. \quad (10)$$

Therefore, a scale-invariant model is not sufficient in itself to ensure that a proper correlation structure will be obtained. Nonetheless, if an invariant model is analyzed, it is always possible by Equation 3 to compute rescaled estimates $\hat{\gamma}^*$ such that Condition 10 does result. This obviously suggests that as a general rule, one should rescale the parameter estimates, if the situation so requires, to ensure that the necessary condition for a

correlation structure holds (Browne, 1982; Krane & McDonald, 1978, section 5).

Krane and McDonald (1978) proved that two classes of invariant factor analysis models give a reproduced matrix with unit diagonal elements when applied to sample correlations (see also Howe, 1955, sections 2.4, 5.2). These are (a) the unrestricted orthogonal model estimated with maximum likelihood and (b) the restricted oblique model (Equation 1), also estimated with maximum likelihood, where $\text{diag}(\Phi) = \mathbf{I}$, Ψ is unconstrained, and restrictions to Λ are only fixed zeros. No corresponding proof could be developed for the restricted orthogonal model under maximum likelihood, even though this model is scale-invariant when Λ contains only fixed zeros. By contrast, there apparently is no class of factor analysis model estimated by generalized least squares that will always satisfy Restriction 10 when applied to correlations, nor are there any broad classes of more general covariance structures that do. In all of these cases, researchers must determine whether estimates from a particular model need to be rescaled before reporting the results as a correlation structure.

To illustrate these ideas, consider a model for the artificial data set shown at the top of Table 1. An unrestricted two-factor orthogonal model was originally fit to these data, which is identified with $df = 4$ if $\lambda_{12} = 0$. As was noted previously, this structure is scale invariant. By Krane and McDonald's (1978) theorem, estimates from the model will satisfy Restriction 10 if it is applied to sample correlations.

Consider a second structure obtained from the first by imposing the additional restriction $\lambda_{61} = 0$. Table 2 contains the maximum likelihood parameter estimates given by LISREL. Also in Table 2 is the reproduced dispersion matrix implied by the estimates. Although this is an assumed correlation structure, it is not the case that $\text{diag}(\hat{\Lambda}\hat{\Lambda}' + \hat{\Psi}) = \mathbf{I}$ from the parameter estimates. The required condition, $\text{diag}(\hat{\Lambda}^*\hat{\Lambda}^{*'} + \hat{\Psi}^*) = \mathbf{I}$, does obtain when a rescaling according to Equations 2 and 3 is applied. In this case, the transformations are $\hat{\Lambda}^* = \mathbf{D}_\alpha\hat{\Lambda}$ and $\hat{\Psi}^* = \mathbf{D}_\alpha^2\hat{\Psi}$, where $\mathbf{D}_\alpha = [\text{diag}(\hat{\Lambda}\hat{\Lambda}' + \hat{\Psi})]^{-1/2}$. These transformed estimates are shown in Table 3, together with the reproduced matrix.

As is explained in the next section, the maximum likelihood test statistic when $\hat{\Lambda}^*$ and $\hat{\Psi}^*$ are obtained from \mathbf{R} will be identical to the test statistic that is obtained when $\hat{\Lambda}$ and $\hat{\Psi}$ are computed from \mathbf{R} . Other derived statistics that use the reproduced matrix may be different, and this could be important in some settings. In this illustration, for example, the root-mean-square residual from the original estimates is 0.141, but that from the rescaled estimates equals 0.126.

In summary, when a model is applied to sample correlations, one should ensure that the reproduced matrix satisfies Restriction 10. If it does not, it should be rescaled according to Equations 2 and 3 so that the requirements of a correlation structure hold.

Effect of Scale Changes on the Omnibus Test Statistic

A second problem that arises in practice concerns the omnibus test statistic associated with the hypothesis

$$H_0: \Sigma = \Sigma(\gamma). \quad (11)$$

Table 1
Data Sets Used in Examples

Artificial data, n not specified ^a							
1.	1.000						
2.	.585	1.000					
3.	.778	.780	1.000				
4.	.155	.368	.281	1.000			
5.	.084	.284	.185	.462	1.000		
6.	.281	.612	.489	.813	.726	1.000	
Vocabulary tests, $n = 649^b$							
1.	15 items, untimed		86.40				
2.	15 items, untimed		57.78	86.26			
3.	75 items, speeded		56.87	59.32	97.29		
4.	75 items, speeded		58.90	59.67	73.82	97.82	
School subjects, $n = 220^c$							
1.	French	1.00					
2.	English	.44	1.00				
3.	History	.41	.35	1.00			
4.	Arithmetic	.29	.35	.16	1.00		
5.	Algebra	.33	.32	.19	.59	1.00	
6.	Geometry	.25	.33	.18	.47	.46	1.00
Aptitude tests for two populations, $n_a = 865, n_b = 900^d$							
1.	63.38				67.90		
2.	70.98	110.24			72.30	107.33	
3.	41.71	52.75	60.85		40.55	55.35	63.20
4.	30.22	37.49	36.39	32.30	28.98	38.90	39.26

^a From "Estimation of Covariance Structure Models When Parameters are Subject to Functional Constraints" by S.-Y. Lee, 1980, *Psychometrika*, 45, p. 316. Copyright 1980 by the Psychometric Society. Adapted by permission.

^b From "Structural Analysis of Covariance and Correlation Matrices" by K. G. Jöreskog, 1978, *Psychometrika*, 43, p. 452. Copyright 1978 by the Psychometric Society. Reprinted by permission.

^c From *An Introduction to Latent Variable Models* (p. 23) by B. S. Everitt, 1984, New York: Chapman and Hall. Copyright by B. S. Everitt. Reprinted by permission.

^d From "Comparison of Correlations, Variances, Covariances, and Regression Weights With and Without Measurement Error" by C. E. Werts, D. A. Rock, R. L. Linn, and K. G. Jöreskog, 1976, *Psychological Bulletin*, 83, p. 1008. Copyright by the American Psychological Association. Reprinted by permission.

The test of this hypothesis depends upon the particular method of estimation used. The method that has received the greatest amount of attention is based upon the maximum likelihood discrepancy function,

$$M = \text{tr}(\mathbf{S}\Sigma^{-1}) - \log |\mathbf{S}\Sigma^{-1}| - p, \tag{12}$$

under the assumption of multivariate normality, from which one obtains maximum likelihood estimates. The arguments to be made, however, also apply to the method of generalized least squares, as well as to other distribution forms. It is well known that the statistic $(n - 1)M$ for evaluating Hypothesis 11 is distributed in large samples as χ^2 , with $df = \frac{1}{2}p(p + 1) - q$, where n is equal to sample size.

When an arbitrary covariance structure is applied to a correlation matrix, the function minimized in general is not Equation 12. Instead, one minimizes a different function,

$$\begin{aligned} \tilde{M} &= \text{tr}(\mathbf{R}\Sigma^{-1}) - \log |\mathbf{R}\Sigma^{-1}| - p = \text{tr}(\mathbf{S}\tilde{\Sigma}^{-1}) - \log |\mathbf{S}\tilde{\Sigma}^{-1}| - p \\ &= \text{tr}[\mathbf{S}(\mathbf{D}_s\mathbf{\Sigma}\mathbf{D}_s)^{-1}] - \log |\mathbf{S}(\mathbf{D}_s\mathbf{\Sigma}\mathbf{D}_s)^{-1}| - p. \end{aligned} \tag{13}$$

Rather obviously, at the respective minima, $\tilde{M} \neq M$, except when $\mathbf{D}_s = \mathbf{I}$. (For a related discussion pertaining to generalized least squares in the context of a specific model, see Lee & Fong, 1983, section 1.) This means that applying a model to \mathbf{S} in general will produce a value of the test statistic different than that obtained from a corresponding analysis of \mathbf{R} . In practical terms, it raises the undesirable possibility that two researchers examining the same model with the same data could reach substantively different conclusions about the plausibility of the model depending only upon the scaling of the sample data.

It also should be noted that because many of the recently proposed alternative indices for assessing the fit of models to data (e.g., Bentler & Bonett, 1980; Browne & Cudeck, in press; Cudeck & Browne, 1983; Steiger & Lind, 1980; Tucker & Lewis, 1973) are themselves functions of the discrepancy index used, this problem in no way is alleviated by emphasizing one of them.

Although this difficulty applies to arbitrary structural models in general, it does not apply to all models. If a structure is scale

Table 2
Parameter Estimates and Reproduced Matrix for a Two-Factor Model,
Without Rescaling for Correlation Structure

Reproduced matrix, unscaled							$\hat{\Lambda}$		$\hat{\Psi}$
1.00							.79	.00 ^a	.38
.45	0.88						.57	.49	.31
.70	.65	0.89					.88	.29	.02
-.11	.33	.13	1.05				-.14	.84	.32
-.16	.26	.05	.68	1.07			-.20	.77	.43
.00	.49	.29	.84	.77	1.00		.00 ^a	1.00	.00 ^b

^a Fixed zero. ^b Constrained boundary value.

invariant, then analogous to Function 13 using the model of Equation 2,

$$\begin{aligned} \tilde{M} &= \text{tr} [\mathbf{R}\Sigma(\gamma^*)^{-1}] - \log |\mathbf{R}\Sigma(\gamma^*)^{-1}| - p \\ &= \text{tr} \{ \mathbf{S}[\mathbf{D}_s \Sigma(\gamma^*) \mathbf{D}_s]^{-1} \} - \log |\mathbf{S}[\mathbf{D}_s \Sigma(\gamma^*) \mathbf{D}_s]^{-1}| - p. \end{aligned} \quad (14)$$

Functions 12 and 14 are not the same, and they have different minimizers. But $\Sigma = \Sigma(\gamma) = \mathbf{D}_s \Sigma(\gamma^*) \mathbf{D}_s$, so at their respective minima, $\tilde{M} = M$. Therefore, one may conduct an analysis with $\mathbf{R} = \mathbf{D}_s^{-1} \mathbf{S} \mathbf{D}_s^{-1}$, with the assurance that the test statistic would be identical to that based on \mathbf{S} (Krane & McDonald, 1978; Lawley & Maxwell, 1971). Obviously, and importantly, any substantive conclusions based upon these tests would also be identical in either type of scaling of the sample data.

As a demonstration of these relationships, consider the covariance matrix in the second section of Table 1. If one applies the factor analysis Model 1 to these data, one obtains $\chi^2 = 0.70$ with both \mathbf{S} and \mathbf{R} . By contrast, if Model 2 is used, one obtains the test statistic $\chi^2 = 5.81$ with sample covariances, but $\chi^2 = 1.50$ with sample correlations. For the purpose of evaluating Hypothesis 11, the correct statistic is the first.

The foregoing discussion suggests a practical rule for determining whether a particular mode is scale invariant. This may be useful if the model is complex and therefore difficult to study algebraically. Fit the structure twice, once to a sample covariance matrix, and again to the matrix of correlations. The structure certainly is not invariant if at the minima, $M \neq \tilde{M}$, or equivalently, if the two test statistics are unequal. If at the minima, $M = \tilde{M}$, the model very likely is invariant, but the equality of the function values is not in itself sufficient evidence.

Effects of Scale Changes on Standard Errors

If Hypothesis 11 cannot be rejected, then it often is of interest to go on to test the significance of individual coefficients of the model using the statistic

$$z_i = \hat{\gamma}_i / \text{se}(\hat{\gamma}_i), \quad (15)$$

where $\text{se}(\hat{\gamma}_i)$ is the estimated standard deviation of γ_i evaluated at $\hat{\gamma}_i$. Let z_c denote the two-tailed critical point of the standard normal distribution at probability α . Then one rejects the hypothesis

$$H_0: \gamma_i = 0$$

at the $100(1 - \alpha)\%$ level when $|z_i| \geq z_c$. In other settings it may be useful to construct confidence intervals of the form

$$P(L_1 \leq \gamma_i \leq L_2) = 1 - \alpha,$$

where $L_1 = \hat{\gamma}_i - z_c \cdot \text{se}(\hat{\gamma}_i)$ and $L_2 = \hat{\gamma}_i + z_c \cdot \text{se}(\hat{\gamma}_i)$. These procedures, although used somewhat less often in practice than the omnibus test, are nonetheless important in many applications of structural models. For example, they constitute the primary evidence for demonstrating that nonzero relationships exist among certain variables in a model.

Most computer programs, following the usual derivation of standard errors (e.g., Lawley & Maxwell, 1971), assume that $\text{se}(\hat{\gamma}_i)$ is estimated from a covariance matrix. When a correlation matrix is used instead, these procedures give incorrect results (de Pijper & Saris, 1982, section 1). Consequently, at least

Table 3
Parameter Estimates and Reproduced Matrix for a Two-Factor Model,
With Rescaling for Correlation Structure

Reproduced matrix, rescaled							$\hat{\Lambda}^*$		$\hat{\Psi}^*$
1.00							.79	.00 ^a	.38
.48	1.00						.61	.52	.36
.74	.74	1.00					.94	.31	.02
-.11	.34	.13	1.00				-.14	.82	.31
-.15	.27	.05	.64	1.00			-.20	.75	.41
.00	.52	.31	.82	.75	1.00		.00 ^a	1.00	.00 ^b

^a Fixed zero. ^b Constrained boundary value.

some standard errors are wrong in virtually all reported analyses of correlation structures.

If a model that is not scale invariant is applied to a correlation matrix with most computer programs, all of the estimated standard errors will be wrong. If a scale-invariant model is applied to a sample correlation matrix, the standard errors associated with scale-free parameters will be correct, but standard errors associated with scale-dependent parameters will be incorrect. Formulas are available that give correct standard errors for scale-invariant models when a correlation matrix is analyzed (see Lawley & Maxwell, 1971, sections 5.3, 7.7, for some factor analysis models estimated by maximum likelihood, and Browne, 1982, section 1.6, for other models and other estimation methods), but as of this writing, these corrections have not been included in most computer programs. In particular, they apparently are not implemented in the current version of LISREL.

For an illustration of the relationships among various standard error estimates, consider Model 3 mentioned earlier. If the model is applied to a sample correlation matrix, then Transformations 3 between original and rescaled parameters are

$$\lambda_i^* = \alpha_i \lambda_i, \quad \psi_i^* = \alpha_i^2 \psi_i, \quad \rho^* = \rho, \quad (16)$$

for $i = 1, \dots, 4$, where in this example, $\alpha_i = 1/s_i$ is the reciprocal of the sample standard deviation of the i th variable. Denote the correct standard error for γ_i^* as $se(\hat{\gamma}_i^*)$. An incorrect standard error for γ_i^* , calculated by most computer programs, is obtained by applying the same transformations to $se(\hat{\gamma}_i)$ as are used in Transformations 16. These would be $\alpha_i \cdot se(\hat{\lambda}_i)$, $\alpha_i^2 \cdot se(\hat{\psi}_i)$, and $se(\hat{\rho})$ for the coefficients λ_i^* , ψ_i^* , and ρ^* , respectively. But with λ_i^* , for example, $se(\hat{\lambda}_i^*) \neq \alpha_i \cdot se(\hat{\lambda}_i)$. So even though $\hat{\lambda}_i^* = \alpha_i \hat{\lambda}_i$, the two intervals,

$$[\hat{\lambda}_i^* - z_c \cdot se(\hat{\lambda}_i^*) \leq \lambda_i^* \leq \hat{\lambda}_i^* + z_c \cdot se(\hat{\lambda}_i^*)]$$

and

$$[\alpha_i \hat{\lambda}_i - z_c \cdot \alpha_i \cdot se(\hat{\lambda}_i) \leq \alpha_i \lambda_i \leq \alpha_i \hat{\lambda}_i + z_c \cdot \alpha_i \cdot se(\hat{\lambda}_i)],$$

are not equal.

For a similar reason, when evaluating

$$H_0: \lambda_i^* = 0, \quad (17)$$

the correct test statistic

$$z_i^* = \hat{\lambda}_i^* / se(\hat{\lambda}_i^*) \quad (18)$$

is not the same as the incorrect ratio

$$z_i = (\alpha_i \hat{\lambda}_i) / [\alpha_i \cdot se(\hat{\lambda}_i)]. \quad (19)$$

Nonetheless, Statistics 19 and 15 are obviously equivalent, so if $\lambda_i = 0$, then $\lambda_i^* = \alpha_i \lambda_i = 0$ as well. Paradoxically, this means that Statistic 19 often can be correctly used to test Hypothesis 17, because whenever $\lambda_i = 0$, then Hypothesis 17 must hold as well.

As an example of the discrepancy that occurs in computing standard errors, consider the correlation matrix in the third section of Table 1. We fit a structure similar to Model 3 but modify the pattern matrix to be

Table 4

Parameter Estimates and Standard Errors, With and Without Corrections for Standardization

Parameter	Estimate	Standard error	
		Uncorrected	Corrected
λ_1	.690	.076	.059
λ_2	.671	.076	.060
λ_3	.531	.076	.064
λ_4	.764	.068	.046
λ_5	.764	.068	.046
λ_6	.616	.069	.053
ϕ_{21}	.598	.072	.072
ψ_1	.524	.082	.082
ψ_2	.550	.082	.080
ψ_3	.718	.082	.080
ψ_4	.416	.069	.070
ψ_5	.416	.069	.070
ψ_6	.620	.072	.065

$$\Lambda' = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & \lambda_5 & \lambda_6 \end{pmatrix}.$$

Again using LISREL and maximum likelihood estimation, we find the omnibus test statistic to be $\chi^2 = 7.92$ with eight degrees of freedom. The estimates of the parameters are listed in column 2 of Table 4. The third and fourth columns of the table list estimated standard errors associated with each parameter, first in their uncorrected form as printed by LISREL and then after corrections for standardization have been applied (Lawley & Maxwell, 1971, section 7.7). The estimated standard error for $\hat{\phi}_{21}$, the only scale-free parameter of this model, is identical in the two sets of coefficients (the value $se(\hat{\psi}_1) = .082$ is also the same, but only coincidentally so). The other standard errors are different in the two columns, at times substantially so. In the worst case, the standard deviation associated with $\hat{\lambda}_4$ and $\hat{\lambda}_5$ is incorrect by a factor of $100(.068 - .046)/.046 = 48\%$.

In summary, if a structural model is applied to a sample correlation matrix using standard error formulas that assume that the covariance matrix has been analyzed, then the values of the estimated standard errors in general are incorrect, unless the parameter is scale-free. This is why $se(\hat{\phi}_{21})$ in the last model is correct in both sets of estimated standard errors. If a computer program has specifically included formulas for standard errors that assume a correlation matrix is to be analyzed, then the standard errors will be correct only when applied to parameter estimates from scale-invariant models (such as the factor analysis Models 1 and 3). If a structure that is not scale invariant (e.g., Model 2) is applied to a sample correlation matrix, then in addition to the incorrect parameter estimates, the standard errors will be incorrect as well.

Rescaling in Multiple Group Analyses

An important development in the study of structural models is the necessary theory for multiple group analyses. This work was pioneered by Jöreskog (1971) for factor analysis and later extended to general covariance structures. The problems asso-

ciated with rescaling observed variables in multiple population studies are potentially more serious than those in studies with single populations, because different transformations can be applied to each group individually. As has often been noted, standardizing covariance matrices separately for each sample removes information about variability that is essential for a correct analysis. Most published studies have been careful to avoid this error.

This does not mean that group covariance matrices must always be used in the analysis of multiple populations. Indeed, for the sake of interpretability of the estimates, it has been recommended that a common rescaling be applied to data from each group (Jöreskog, 1971, section 3.4). In this procedure, one obtains samples of size n_g from $g = 1, \dots, m$ independent populations, most often on a common set of p variables, producing m distinct covariance matrices, S_g . These in turn may be transformed in a consistent manner, using

$$S_g^* = DS_gD, \quad (20)$$

where $D = \text{diag}(S)^{-1/2}$, and $S = (n-1)^{-1} \sum (n_g - 1)S_g$ is the pooled covariance for $n = n_1 + \dots + n_m$ combined cases. It has been thought that this rescaling can be performed at the convenience of the researcher if so desired.

In light of the discussion pertaining to rescaling covariance matrices in a single population, it can be seen that this practice cannot always be followed. In particular, it will only be possible to rescale in this way if each model applied to the m groups is scale invariant. As an example, consider the case in which $m = 2$ and suppose that Model 3 is applied to the first sample and that Model 2 is applied to the second. It can be shown (Jöreskog, 1971, section 2.3) that if there are no constraints between parameters across the various groups, then Function 12, the maximum likelihood discrepancy function, will be the sum of the two discrepancy functions for each model applied separately, that is, $M = M_1 + M_2$. (For brevity in this section, M will denote the minimum of Function 12 over the parameter space.) In the present case, the model applied to Sample 1 is scale invariant, so whether the rescaling in Equation 20 or any other instance of Equation 2 is used, one has on the basis of Equation 14 $\tilde{M}_1 = M_1$. In contrast, the model for Sample 2 is scale dependent, so in general, $M_2 \neq \tilde{M}_2$. By reasoning in this way, it can be shown that in any analysis in which one or more of the models applied to the m populations is not scale invariant, it must be that $M \neq \tilde{M}$ for the overall discrepancy function.

For example, consider the data from a two-population study given in the bottom section of Table 1. In Table 5 are results from applying several combinations of the factor analysis Models 2 and 3, either to sample covariance matrices or to the matrices rescaled as in Equation 20. In the first analysis, Model 3 is applied to each sample, giving identical values of the test statistic for either kind of scaling. In the second analysis, Model 2 is applied to both groups. This model is not scale invariant, and the test statistics are quite different for the matrices S_g in comparison with matrices S_g^* . The third analysis applies Model 3 to Sample 1 and Model 2 to Sample 2. Again, the test statistics are very different. The final analysis fits Model 3 to each sample and also constrains the factor-loading matrices to be equal in

Table 5
Test Statistics for Models Applied to Two Samples, With Scaled or Unscaled Covariance Matrices

Analysis	Model		Test statistic for		df
	Group 1	Group 2	S_g	S_g^*	
1	3	3	2.18	2.18	2
2	2	2	145.57	2.20	4
3	3	2	81.61	2.19	3
4	3 ^a	3 ^a	7.65	7.65	6

^a Equality constraints between factor pattern matrices.

the two populations, so that $\Lambda_1 = \Lambda_2$. Here again the test statistics are equal. It is interesting to note how similar the pattern of test statistics is in Analyses 1, 2, and 3 using rescaled covariance matrices when, as can be seen with the unscaled dispersion matrices, the performance of the various combinations of models is actually very different.

Alternatives for Analyses of Correlation Matrices

In any study of covariance structures, a complete statistical analysis gives estimates of model parameters, estimates of standard errors for these parameters, and a test of fit for the whole model, in practice usually based on either the maximum likelihood or generalized least squares test statistics. When a model that is not scale invariant is applied to a sample correlation matrix using a program designed for covariance structure analysis such as the current version of LISREL, all of these computed quantities are incorrect, and the model studied is actually modified with respect to the model originally intended, such as was illustrated in Equation 9. An obvious protection against these errors is to always conduct an analysis using the sample covariance matrix, for then none of these problems can occur. In many research reports, there appears to be no compelling reason why the authors prefer a correlation structure rather than a covariance structure, except perhaps the habit associated with using correlations.

Of course in other settings, the analysis of standardized variables is highly desirable in its own right (Kim & Feree, 1981). If a scale-invariant model is applied to sample correlations, then the structure is not modified, and the omnibus test statistic is accurate. Furthermore, estimates of model parameters are correct, subject perhaps only to a final rescaling to ensure that Restriction 10 holds. The standard errors produced by most computer programs will still be incorrect for some parameters, and there appears to be no way in general to avoid this aspect of the problem without using special procedures. In many instances, a structure may contain one or more scale-free parameters. The standard errors for these coefficients will be correct, even if other parameters in the model are scale dependent.

In many investigations of correlation structures, the model is not scale invariant, and one cannot simply proceed in the usual manner. The factor analysis Model 2 is an example of such a structure. In this case, the only possible alternative is to employ

a completely different methodology that will yield correct results when applied to a correlation matrix. The general strategy is to estimate and test correlation structures in the context of models for covariance matrices (Krane & McDonald, 1978):

$$\Sigma = \Sigma(\gamma) = \Delta P(\theta)\Delta, \quad (21)$$

where the diagonal matrix Δ contains scaling terms δ_i and the correlation structure $P = P(\theta)$ satisfies the restrictions

$$\text{diag}(P) = I. \quad (22)$$

The parameter vector of the covariance structure then is $\gamma' = (\delta_1, \dots, \delta_p, \theta')$.

Two general approaches have been used to estimate the correlation structure $P(\theta)$ while ensuring that Restriction 22 holds. In the first case, reparameterizations of the form

$$\Sigma = \Delta\{\Gamma(\gamma) + I - \text{diag}[\Gamma(\gamma)]\}\Delta \quad (23)$$

can be considered. For example, with factor analysis, one writes

$$\Sigma = \Delta[\Lambda\Phi\Lambda' + I - \text{diag}(\Lambda\Phi\Lambda')]\Delta$$

(Browne & du Toit, 1987, section 6.2; Jennrich, 1974, section 1). The uniquenesses corresponding to Equation 1 are implicitly given by the difference $\Psi = I - \text{diag}(\Lambda\Phi\Lambda')$. Alternatively, Equation 22 can be imposed directly using numerical methods (Bentler & Lee, 1983; Browne, 1982). Then Δ requires estimating p additional parameters, but imposing the restriction of Equation 22 numerically provides p additional constraints, so that the degrees of freedom of the model are actually unaffected. Although Model 23 appears to be the easier of the two approaches to implement in a computer program, Heywood cases will occur whenever elements of $\text{diag}[\Gamma(\gamma)]$ are greater than unity. If Restriction 22 is imposed numerically, Heywood cases can also occur, although a method for imposing inequality constraints could be applied to restrict all $\psi_i \geq 0$.

Model 21 has the important property that it is scale invariant. It also has the property that the parameters in the correlation structure $P(\theta)$ are scale-free. Therefore all of the problems reviewed in this article can be avoided, except for the matter of standard errors for Δ . Because Δ contains scaling terms, however, it is generally uninteresting to estimate these parameters without information about the variances of the measures. The overriding motivation for studying correlation structures appears to be the problem of interpretation. Therefore an advantage of structures such as Model 21 is that they may be applied to covariance matrices but retain the feature that the parameter matrices of the function $P(\theta)$ are in the standardized metric of correlations.

One drawback with this approach is that computer programs to estimate Model 21 under Restriction 22 are not yet widely available. LISREL,² in particular, cannot yet handle the restrictions needed for factor analysis, namely,

$$\sum_{j=1}^f \sum_{k=1}^f \lambda_{ij} \lambda_{ik} \phi_{jk} + \psi_{ii} = 1,$$

for $i = 1, \dots, p$. Nonetheless, two computer programs, BMDP's PAR (Lee & Jennrich, 1984) and COSAN (McDonald, 1980), can be used to fit Model 23. Browne and du Toit's (1987) AUFIT

can impose Restriction 22, either directly using numerical methods or by the reparameterization of Equation 23.

A reviewer of this article suggested that Model 21 in conjunction with Restriction 22 be generally recommended in covariance structure analysis, especially for the analysis of correlation structures. Although this approach is frequently useful, it may not be optimal, or even desirable, for all models. In the first case, imposing Restriction 22 as a nonlinear numerical constraint adds appreciably to the computing costs, especially when the model is complex or when a large number of variables is involved. Similarly, it can be time-consuming to operationalize Model 23 in special-purpose computer programs if many models are to be studied. In these cases, scale-invariant models are attractive, for they can be estimated with programs that employ traditional algorithms.

A second limitation of Model 23 or 21 with Restriction 22 as general models for covariance structures is that these formulations simply will not always be appropriate. Consider, for example, a model for multitrait-multimethod covariance matrices (Browne, 1984):

$$\Sigma = \Delta(P_m \otimes P_t + D_u^2)\Delta, \quad (24)$$

where P_m and P_t represent correlation matrices among method and trait common scores, respectively, and D_u^2 and Δ are diagonal. The direct product between the component correlation matrices, $P_m \otimes P_t$, itself yields a correlation matrix, so that $\text{diag}(P_m \otimes P_t) = I$. Elements of D_u^2 are nonnegative, however, so diagonal elements of $P_m \otimes P_t + D_u^2$ in general are greater than unity. Therefore Model 24 is not a specialization of Model 21 or 23. Model 24 is scale invariant, and the parameters P_m , P_t , and D_u are scale-free, so a standard algorithm can be used to solve for the parameters.

Several other models have been used that also are fundamentally inappropriate as correlation structures. Examples are parallel or tau-equivalent test models and certain longitudinal or growth models. Because these structures specify functions of variances, there is no way to translate them into corresponding correlation structures.

Discussion

Statistical theory for the analysis of covariance structures, as well as the computer programs that implement them, has been most completely developed for applications to a sample covariance matrix. When variables have quite different variances, however, it is useful, at times even necessary, to standardize the variables and to carry out the analysis on the matrix of sample correlations to facilitate comparisons among the coefficients associated with different tests. In psychology, this is common

² A reviewer also recommended a LISREL reparameterization scheme suggested by Rindskopf (1984) for factor analysis models to constrain the diagonal of a reproduced covariance matrix to unities. This method is clever but limited in practice, because it applies only to the special case of models in which each variable is regressed on a single factor. Even in these cases, the standard errors for uniquenesses are not available. Models of this kind are scale invariant, however, and generally can be handled by rescaling as described in the text.

practice when fitting the regression model or using traditional methods of factor analysis. But it is unjustifiable as a general strategy for analyzing structural models, because it often leads to inferential and interpretive errors. By analyzing a correlation matrix, one may (a) implicitly alter the model being studied, (b) produce a value of the omnibus test statistic that is incorrect, or (c) report standard errors that are quite discrepant from the correct values. In comparison with the analysis of a covariance matrix, the corresponding analysis of a correlation matrix should be viewed as a special case, one that always requires justification.

With scale-invariant models, Errors a and b cannot occur. Even when a model is scale invariant, Error c will occur for all parameters that are not scale-free and that have not been estimated by special procedures developed for this circumstance.

Recent research has led to general procedures that are appropriate for analyzing virtually any structural model with a matrix of correlations (Browne & du Toit, 1987). Computer programs for this method have not yet been widely distributed, however, and the number of applications of the method is small.

It may be useful to provide general guidelines for scale-invariant factor analysis models. More complete discussions of specific models are given in Browne (1982), Krane and McDonald (1978), and Swaminathan and Algina (1978). It should be noted that defining a model to ensure that it is scale invariant may interact with the model's identifiability. Prudence requires that each of these matters be investigated on a case-by-case basis.

If the factor covariance matrix is constrained in $\mathbf{I} = \text{diag}(\Phi)$, if Ψ is unconstrained, and if only fixed values of zero are introduced to Λ , then one obtains a scale-invariant model. This class of structures subsumes congeneric test models but not tau-equivalent or parallel test models.

If Ψ is unconstrained, and if Φ is also unconstrained so that $\text{diag}(\Phi) \neq \mathbf{I}$, then models with a single fixed nonzero element in each column of Λ are scale invariant. Combining both of these classes shows that many models for latent variable regression are scale invariant. For example, if latent independent variables are specified according to guidelines for the first class and latent dependent variables are determined according to the second class, then the resulting model is scale invariant (e.g., Maruyama & McGarvey, 1980).

Many factor analysis models are not invariant with respect to rescaling. For example, if the constraint $\text{diag}(\Phi) = \mathbf{I}$ is imposed, and if additional fixed nonzero values are specified in Λ , then the model is not scale invariant. If a model imposes two or more nonzero fixed values in a single column of Λ , then it is not scale invariant. Similarly, if two or more elements are constrained to be equal in one column of Λ , or if two or more elements are constrained to be equal in different columns of Λ , or if two or more elements of Ψ are constrained to be equal, then the model is not scale invariant.

References

- Bagozzi, R. P. (1980). *Causal models in marketing*. New York: Wiley.
- Bentler, P. M. (1983). Simultaneous equation systems as moment structure models. *Journal of Econometrics*, 22, 13–42.
- Bentler, P. M., & Bonett, D. G. (1980). Significance tests and goodness of fit in the analysis of covariance structures. *Psychological Bulletin*, 88, 588–606.
- Bentler, P. M., & Lee, S.-Y. (1978). Statistical aspects of a three-mode factor analysis model. *Psychometrika*, 43, 343–352.
- Bentler, P. M., & Lee, S.-Y. (1983). Covariance structures under polynomial constraints: Applications to correlation and alpha-type structural models. *Journal of Educational Statistics*, 8, 207–222.
- Blok, H., & Saris, W. E. (1983). Using longitudinal data to estimate reliability. *Applied Psychological Measurement*, 7, 295–301.
- Browne, M. W. (1974). Generalized least squares estimators in the analysis of covariance structures. *South African Statistical Journal*, 8, 1–24.
- Browne, M. W. (1977). The analysis of patterned correlation matrices by generalized least squares. *British Journal of Mathematical and Statistical Psychology*, 30, 113–124.
- Browne, M. W. (1982). Covariance structures. In D. M. Hawkins (Ed.), *Topics in applied multivariate analysis* (pp. 72–141). Cambridge, England: Cambridge University Press.
- Browne, M. W. (1984). The decomposition of multitrait-multimethod matrices. *British Journal of Mathematical and Statistical Psychology*, 37, 1–21.
- Browne, M. W., & Cudeck, R. (in press). Single sample cross-validation indices for covariance structures. *Multivariate Behavioral Research*.
- Browne, M. W., & du Toit, S. H. C. (1987). *Automated fitting of nonstandard models* (Report No. WS-39). Pretoria, South Africa: Human Sciences Research Council.
- Cole, D. A., Howard, G. S., & Maxwell, S. E. (1981). Effects of mono-versus multiple-operationalization in construct validation efforts. *Journal of Consulting and Clinical Psychology*, 49, 395–405.
- Cornelius, S. W., Willis, S. L., Nesselroade, J. R., & Baltes, P. B. (1983). Convergence between attention variables and factors of psychometric intelligence in older adults. *Intelligence*, 7, 253–269.
- Cudeck, R., & Browne, M. W. (1983). Cross validation of covariance structures. *Multivariate Behavioral Research*, 18, 147–167.
- Cunningham, W. R. (1980). Age comparative factor analysis of ability variables in adulthood and old age. *Intelligence*, 4, 133–149.
- de Pijper, W. M., & Saris, W. E. (1982). The effect of identification restrictions on the test statistic in latent variable models. In K. G. Jöreskog & H. Wold (Eds.), *Systems under indirect observation: Causality, structure, prediction: Part 1* (pp. 175–182). New York: North-Holland.
- Everitt, B. S. (1984). *An introduction to latent variable models*. New York: Chapman and Hall.
- Howe, W. G. (1955). *Some contributions to factor analysis*. (Research Report No. 1919). Oak Ridge, TN: Oak Ridge National Laboratory.
- Jennrich, R. I. (1974). Simplified formulae for standard errors in maximum-likelihood factor analysis. *British Journal of Mathematical and Statistical Psychology*, 27, 122–131.
- Jöreskog, K. G. (1970). A general method for analysis of covariance structures. *Biometrika*, 57, 239–251.
- Jöreskog, K. G. (1971). Simultaneous factor analysis in several populations. *Psychometrika*, 36, 409–426.
- Jöreskog, K. G. (1978). Structural analysis of covariance and correlation matrices. *Psychometrika*, 43, 443–477.
- Jöreskog, K. G. (1981). Analysis of covariance structures. *Scandinavian Journal of Statistics*, 8, 65–92.
- Jöreskog, K. G., & Sörbom, D. (1977). Statistical models and methods for analysis of longitudinal data. In D. J. Aigner & A. S. Goldberger (Eds.), *Latent variables in socioeconomic models* (pp. 285–325). Amsterdam: North-Holland.
- Jöreskog, K. G., & Sörbom, D. (1984). *LISREL VI User's Guide*. Mooresville, IN: Scientific Software, Inc.

- Kim, J.-O., & Feree, G. D. (1981). Standardization in causal analysis. *Sociological Methods and Research*, 10, 187-210.
- Krane, W. R., & McDonald, R. P. (1978). Scale invariance and the factor analysis of covariance matrices. *British Journal of Mathematical and Statistical Psychology*, 31, 218-228.
- Lansman, M., Donaldson, G., Hunt, E., & Yantis, S. (1982). Ability factors and cognitive processes. *Intelligence*, 6, 347-386.
- Lawley, D. N., & Maxwell, A. E. (1971). *Factor analysis as a statistical method*. London: Butterworth.
- Lee, S.-Y. (1979). Constrained estimation in covariance structure analysis. *Biometrika*, 66, 539-545.
- Lee, S.-Y. (1980). Estimation of covariance structure models when parameters are subject to functional constraints. *Psychometrika*, 45, 309-324.
- Lee, S.-Y., & Fong, W.-K. (1983). A scale invariant model for three-mode factor analysis. *British Journal of Mathematical and Statistical Psychology*, 36, 217-223.
- Lee, S.-Y., & Jennrich, R. I. (1984). The analysis of structural equation models by means of derivative-free nonlinear least squares. *Psychometrika*, 49, 521-528.
- Linn, R. L., & Werts, C. E. (1982). Measurement error in regression. In G. Keren (Ed.), *Statistical and methodological issues in psychology and social science research* (pp. 131-154). Hillsdale, NJ: Erlbaum.
- MacLeod, C. M., Jackson, R. A., & Palmer, J. (1986). On the relation between spatial ability and field dependence. *Intelligence*, 10, 141-151.
- Marsh, H. W., & Hocevar, D. (1983). Confirmatory factor analysis of multitrait-multimethod matrices. *Journal of Educational Measurement*, 20, 231-248.
- Maruyama, G., & McGarvey, B. (1980). Evaluating causal models: An application of maximum-likelihood analysis of structural equations. *Psychological Bulletin*, 87, 502-512.
- McDonald, R. P. (1975). Testing pattern hypotheses for correlation matrices. *Psychometrika*, 40, 189-201.
- McDonald, R. P. (1980). A simple comprehensive model for the analysis of covariance structures: Some remarks on applications. *British Journal of Mathematical and Statistical Psychology*, 33, 161-183.
- Rindskopf, D. (1984). Using phantom and imaginary latent variables to parameterize constraints in linear structural models. *Psychometrika*, 49, 37-47.
- Shapiro, A., & Browne, M. W. (1986). *On the treatment of correlation structures as covariance structures* (Research Report No. 86/2). Pretoria, South Africa: University of South Africa, Department of Statistics.
- Steiger, J. H., & Lind, J. C. (1980, May). *Statistically-based tests for the number of common factors*. Paper presented at the annual meeting of the Psychometric Society, Iowa City, IA.
- Swain, A. J. (1975). A class of factor analysis estimation procedures with common asymptotic sampling properties. *Psychometrika*, 40, 315-335.
- Swaminathan, H., & Algina, J. (1978). Scale freeness in factor analysis. *Psychometrika*, 43, 581-583.
- Tucker, L. R., & Lewis, C. A. (1973). A reliability coefficient for maximum likelihood factor analysis. *Psychometrika*, 38, 1-10.
- Werts, C. E., Breland, H. M., Grandy, J., & Rock, D. R. (1980). Using longitudinal data to estimate reliability in the presence of correlated measurement errors. *Educational and Psychological Measurement*, 40, 19-29.
- Werts, C. E., Rock, D. A., Linn, R. L., & Jöreskog, K. G. (1976). Comparison of correlations, variances, covariances, and regression weights with and without measurement error. *Psychological Bulletin*, 83, 1007-1013.

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Hunt Appointed Editor of *JEP: General*, 1990-1995

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